Lecture 9 - Complex Dynamics

Local Fixed point theory

Def.: If z_0 s.t. $f(z_0) = z_0$ then the **multiplier** is

$$\lambda := f'(z_0)$$

① if $|\lambda| < 1$, z_0 is ATTRACTING
  if $|\lambda| = 0$, z_0 is SUPERATTRACTING
  } Fatou set

② if $|\lambda| > 1$, z_0 is REPELLING } Julia set

③ if $|\lambda| = 1$, z_0 is INDIFFERENT
  if $\exists q : \lambda^q = 1$, z_0 is PARABOLIC } Julia set
  if $\lambda = e^{2\pi i\alpha}$, $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, z_0 is IRRATIONALLY INDIFFERENT

$z \mapsto e^{2\pi i\alpha} z$

if they belong to Fatou set, they are centers of SIEGEL DISKS

if they belong to Julia set, they are called CREMER POINTS

Consider $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, $\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}}$ CONTINUED FRACTION

$$\frac{p_n}{q_n} = [a_1, a_2, \dots, a_n]$$

$$B(\alpha) := \sum_{n=0}^{\infty} \frac{\log q_{n+1}}{q_n}$$

α is a BRJUNO NUMBER if $B(\alpha) < \infty$.

Thm (Yoccoz)

α is a Brjuno number $\longleftrightarrow$ every holo function with multiplier $e^{2\pi i\alpha}$ has a Siegel disk.

E.g.: $\alpha = \dfrac{\sqrt{5}-1}{2} = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\dots}}}$

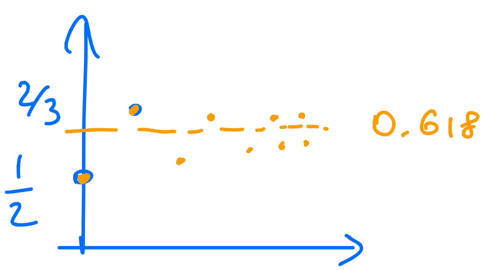
$$\alpha = \frac{1}{1+\alpha} \longleftrightarrow \alpha + \alpha^2 = 1$$

$$\alpha^2 + \alpha - 1 = 0$$

$$\alpha = \frac{-1 \pm \sqrt{5}}{2}$$

$$\frac{p_2}{q_2} = \frac{1}{1+\frac{1}{1}} = \frac{1}{2} \qquad q_2 = 2$$

$$\frac{p_3}{q_3} = \cfrac{1}{1+\cfrac{1}{1+\frac{1}{1}}} = \frac{2}{3}$$



$$\frac{3}{5},\ \frac{5}{8},\ \frac{8}{13},\ \frac{13}{21},\ \dots \longrightarrow \alpha = \frac{\sqrt{5}-1}{2}$$

$$q_{n+1} = a_{n+1} q_n + q_{n-1} = q_n + q_{n-1} \leq 2q_n$$

$$\sum_{n=0}^{\infty} \frac{\log q_{n+1}}{q_n} \leq \sum_{n=0}^{\infty} \frac{\log(2q_n)}{q_n} \leq \sum_{n=0}^{\infty} \frac{\log 2 + \log q_n}{q_n}$$

$$q_{n+1} = q_n + q_{n-1} \geq q_n + \frac{1}{2} q_n \geq \frac{3}{2} q_n$$

$$\frac{1}{2} \leq \frac{q_{n-1}}{q_n} \leq \frac{2}{3} \qquad \Longrightarrow \quad q_n \geq \left(\frac{3}{2}\right)^n$$

$$\frac{\log(2q_n)}{q_n} \leq \frac{1}{q_n^{1-\varepsilon}} \leq \frac{1}{\left(\frac{3}{2}\right)^{n(1-\varepsilon)}}$$

hence $\displaystyle\sum_{n=0}^{\infty} \frac{\log q_{n+1}}{q_n} < \infty \Longrightarrow \alpha = \frac{\sqrt{5}-1}{2}$ is <u>Brjuno</u>

$$f(z) = e^{2\pi i \frac{\sqrt{5}-1}{2}} z + z^2 + 100 z^{199} + 2^{10} z^{17^5}$$

has a Siegel disk containing 0.

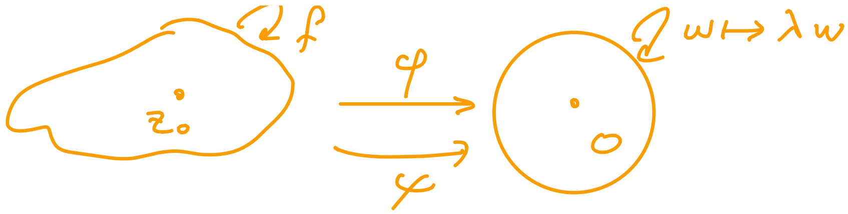
<u>Linearization of attracting fixed points</u>

<u>Thm</u> (Königs)

If $f(z_0) = z_0$ and $\lambda = f'(z_0)$, if $|\lambda| \neq 0, 1$, then $\exists\ \varphi$ a holo function defined over some disk s.t. $\varphi(0) = 0$

$$\varphi \circ f \circ \varphi^{-1}(w) = \lambda w$$

for all $w \in \mathbb{D}_r$. Moreover φ is unique up to multiplication by a constant.



Proof: ① uniqueness

if there are two such maps φ, ψ

$$u = \psi \circ \varphi^{-1}(w) = b_1 w + b_2 w^2 + \dots + b_n w^n + \dots$$

$$\begin{tikzcd}
\mathcal{U} \arrow[loop above, "f"] \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "\psi"'] & \mathbb{D}_r \arrow[loop above, "\lambda^*"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{U} \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "\psi"'] \arrow[r, bend left=60, "u"] \arrow[d, "f"'] & \mathbb{D}_r \arrow[d, "\lambda^*"] \\
\mathcal{U} \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "\psi"'] & \mathbb{D}_r \arrow[loop left, "u"]
\end{tikzcd}$$

$$u(\lambda w) = \lambda u(w)$$

$$\varphi \circ f \circ \varphi^{-1}(w) = \lambda w$$

$$\underbrace{u \circ \varphi} \circ f \circ \underbrace{\varphi^{-1} \circ u^{-1}}(z) = u(\lambda u^{-1}(z))$$

$$\varphi \circ f \circ \varphi^{-1}(z) = \lambda z$$

$\lambda b_n = b_n \lambda^n$ for every $n \geq 1$,

$\lambda \neq 0$, $|\lambda| \neq 1$

$\Longrightarrow$ $b_n = 0$ for all $n \geq 2$.

$u(z) = b_1 z \Longrightarrow \varphi$ is unique up to multiplication.

② <u>Existence</u> Need to find φ

$$\varphi \circ f(z) = \lambda \varphi(z)$$

Let $0 < |\lambda| < 1$. Choose $c < 1$ with $c^2 < |\lambda| < c$. and a nbd D_r of 0 s.t. $|f(z)| \leq c|z|$ on D_r. For any $z_0 \in D_r$, $z_n := f^n(z_0) \xrightarrow[\text{as } n \to \infty]{} 0$.

$|z_n| \leq r c^n$. By Taylor expansion, $\exists\, k$ s.t.

$$|f(z) - \lambda z| \leq k |z|^2 \quad \text{for } z \in D_r.$$

$$z_{n+1} = f^{n+1}(z_0) = f(z_n)$$

$$|f(z_n) - \lambda z_n| \leq k|z_n|^2 \leq k r^2 c^{2n}$$

$$\|$$

$$|z_{n+1} - \lambda z_n|$$

Consider $\quad w_n := \dfrac{z_n}{\lambda^n}$

$$|w_{n+1} - w_n| = \left| \frac{z_{n+1}}{\lambda^{n+1}} - \frac{z_n}{\lambda^n} \right| = \frac{|z_{n+1} - \lambda z_n|}{|\lambda|^{n+1}} \leq$$

$$\leq \frac{k r^2 c^{2n}}{|\lambda|^{n+1}} = \frac{k r^2}{|\lambda|} \left(\frac{c^2}{|\lambda|} \right)^n$$

Hence $\lim\limits_{n\to\infty} w_n$ converges geometrically

$\varphi(z) := \lim\limits_{n\to\infty} \dfrac{f^n(z)}{\lambda^n}$ converges uniformly on compact subsets

Check $\quad \varphi(f(z)) \overset{?}{=} \lambda \varphi(z)$

$$\varphi(f(z)) = \lim_{n\to\infty} \frac{f^n(f(z))}{\lambda^n} = \lim_{n\to\infty} \frac{f^{n+1}(z)}{\lambda^{n+1}} \lambda$$

$$= \lambda \varphi(z) \quad \boxtimes$$

Cor.: If $|\lambda| > 1$, there is a local linearizing map φ, by considering f^{-1}.

Cor.: if z_0 is repelling fixed pt, for any other z_1 in $J(f)$, $f^{-n}(z_1) \in J(f)$

If λ is not real, then $(f^{n}(z_1))$ lie approximately on a logarithmic spiral.

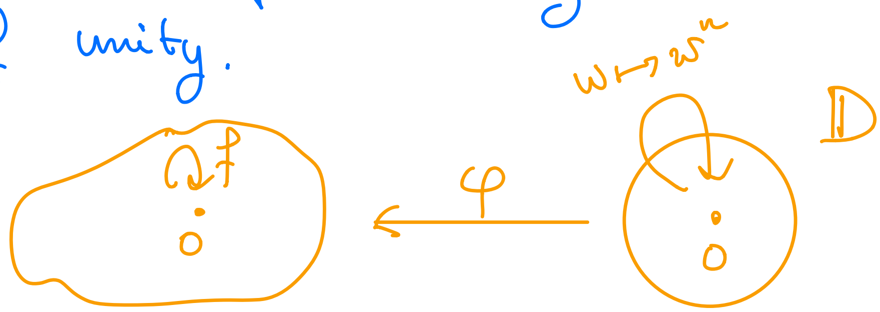
So, $J(f)$ is NOT a smooth manifold in a nbd of z_0.

Thm (Böttcher)

Suppose that

$$f(z) = a_n z^n + a_{n+1} z^{n+1} + \cdots$$

with $a_n \neq 0$, $n \geq 2$. Then there exists a local holomorphic change of coords $w = \phi(z)$ which conjugates f to $w \mapsto w^n$ s.t. $\varphi(0) = 0$. Moreover, φ is unique up to multiplication by a $(n-1)$th root of unity.



Proof if such φ exists:

$$\varphi(w^n) = f(\varphi(w))$$

By iterating k times:

$$\varphi(w^{n^k}) = f^k(\varphi(w))$$

$$\varphi(z) :\overset{?}{=} \sqrt[n^k]{f^k(z)}$$

<u>More formally</u>: assume $z_0 = \infty$

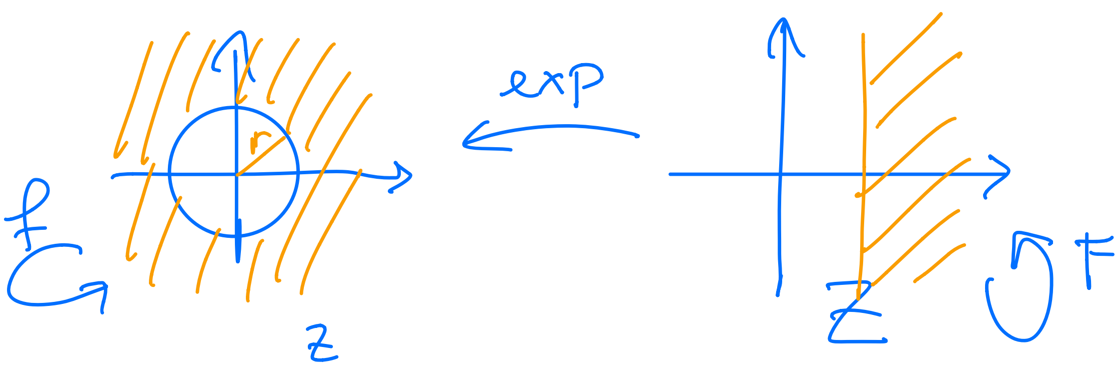
$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_0 + a_{-1} z^{-1} + \dots$$

$n \geq 2$, convergent for $|z| > r$

By conjugating it by $z \mapsto \alpha z$, assume $a_n = 1$

$$f(z) = z^n\left(1 + O\left(\frac{1}{|z|}\right)\right)$$

$z = e^Z$, $\quad \mathrm{Re}(Z) > \log(r)$



lift f to F s.t.

$$F(Z) = \log f(e^Z)$$

$$
\begin{tikzcd}
\mathrm{Re}(z) > \log r \arrow[r, "F"] \arrow[d, "\exp"'] & \mathrm{Re}(z) > \log r \arrow[d, "\exp"] \\
\mathbb{C} \setminus \overline{\mathbb{D}}_r \arrow[r, "f"] & \mathbb{C} \setminus \overline{\mathbb{D}}_r
\end{tikzcd}
$$

$$|F(Z) - nZ| = O\left(e^{-\mathrm{Re}(Z)}\right) < 1. \quad (*)$$

$\sigma > 1$ s.t. $(*)$ holds for $\mathrm{Re}(Z) > \sigma$.

if $Z_k = F^k(Z_0)$,

$$|Z_{k+1} - nZ_k| < 1$$

Set $W_k = \dfrac{Z_k}{n^k}$

$$|W_{k+1} - W_k| = \left| \frac{Z_{k+1}}{n^{k+1}} - \frac{Z_k}{n^k} \right| =$$

$$= \frac{|Z_{k+1} - nZ_k|}{n^{k+1}} \leq \frac{1}{n^{k+1}}$$

Define $\Phi(Z) := \lim\limits_{k \to \infty} W_k(Z)$

$$= \lim_{k\to\infty} \frac{F^k(z)}{n^k}$$

$$\Longrightarrow \quad \Phi(F(z)) = n\,\Phi(z)$$

$$\Phi(z + 2\pi i) = \Phi(z) + 2\pi i$$

$\Longrightarrow$ Define $\varphi(z) := \exp\left(\Phi(\log z)\right)$

satisfies $\quad \varphi(f(z)) = \varphi(z)^n \quad$ $(**)$

<u>Uniqueness</u> if φ, ψ satisfy $(**)$,

then $u = \varphi \circ \psi^{-1}$ satisfies

$$u(w^n) = (u(w))^n$$

$$u(w) = c_1 w + c_2 w^2 + \dots + c_k w^k + \dots$$

$$u(w^n) = c_1 w^n + c_2 w^{2n} + \dots + c_k w^{nk} + \dots$$

$$(u(w))^n = (c_1 w + c_2 w^2 + \dots)^n$$

$$= c_1^n w^n + \dots.$$

$\Longrightarrow$ $c_1^n = c_1$ $\Longrightarrow$ c_1 is $(n-1)$ root of 1.

$\Longrightarrow$ by induction, all $c_n = 0$ for $n \geq 2$.

E.g.: $n = 4$

$$u(w) = \xi w = e^{\frac{2\pi i}{3}} w$$

$$u(w^4) = \xi w^4 = e^{2\pi i/3} w$$

$$(u(w))^4 = \xi^4 w^4 = e^{4 \cdot 2\pi i/3} w$$

$$= e^{2\pi i/3} w$$

Cor.: Consider Multibrot set

$M_d = \{ c \in \mathbb{C}$ s.t. 0 has bounded orbit under $z \mapsto z^d + c \}$.